Nonrelativistic field theoretic setting for gravitational self-interactions.

Sergio De Filippo*

Dipartimento di Fisica "E. R. Caianiello", Università di Salerno

Via Allende I-84081 Baronissi (SA) ITALY

Tel: +39 089 965 229, Fax: +39 089 965 275, e-mail: defilippo@sa.infn.it

and

Unità INFM Salerno

(February 7, 2008)

It is shown that a recently proposed model for the gravitational interaction in non relativistic quantum mechanics is the instantaneous action at a distance limit of a field theoretic model containing a negative energy field. It reduces to the Schroedinger-Newton theory in a suitable mean field approximation. While both the exact model and its approximation lead to estimates for localization lengths, only the former gives rise to an explicit non unitary dynamics accounting for the emergence of the classical behavior of macroscopic bodies.

04.60.-m 03.65.Ta

In a recent paper [1] a model for the gravitational interaction in nonrelativistic quantum mechanics was proposed. In it matter degrees of freedom were duplicated and gravitational interactions were introduced between observable and unobservable degrees of freedom only. The non unitary dynamics one is led to, once unobservable degrees of freedom are traced out, includes both the traditional aspects of classical gravitational interactions and a form of fundamental decoherence, which may be connected with the emergence of the classical behavior of macroscopic bodies. The model in fact treats on an equal footing mutual and self-interactions, which, by some authors, are possibly held responsible for wave function localization and/or reduction [2–6].

Interactions between observable and unobservable degrees of freedom are instantaneous action at a distance ones and at first sight it looks unlikely that they can be obtained as a nonrelativistic limit of more familiar local interactions mediated by quantized fields. If, in fact, a local interaction were introduced between an ordinary field and observable and unobservable matter, the corresponding low energy limit would include an action at a distance inside the observable (and the unobservable) matter too. In this letter we want to show that a field theoretic reading of the model emerges naturally through a Stratonovich-Hubbard transformation [7] of the gravitational interaction. Within the minimal possible generalization of the model, where the instantaneous interaction is replaced by a retarded potential, the result of the transformation corresponds to the emergence of an ordinary scalar field and a negative energy one, both coupled with the matter through a Yukawa interaction. This result has a plain physical reading, as it gives rise to two competing interactions with overall vanishing effect within observable (and unobservable) matter: an attractive and a repulsive one respectively mediated by the positive and the negative energy field.

The presence of a negative energy field somehow is not surprising if we consider that, in a dynamical theory supposed to account for wave function localization, one expects a small continuous energy injection [10,11], and negative energy fields are the most natural candidates for that, apart from the possible introduction of a cosmological background. In fact the possible role of negative-energy fields was already suggested within some attempts to account for wave function collapse by phenomenological stochastic models [12].

As a by-product of the Stratonovich-Hubbard transformation we show also that a proper mean field approximation leads to the Schroedinger-Newton (S-N) model [6]. Finally, while the original model and the S-N approximation are equivalent as to the classical aspects of the gravitational interaction, the S-N model is shown to be ineffective in turning off quantum coherences corresponding to different locations of one and the same macroscopic body, at variance with the original model.

To be specific, following Ref. [1], let $H[\psi^\dagger, \psi]$ denote the second quantized non-relativistic Hamiltonian of a finite number of particle species, like electrons, nuclei, ions, atoms and/or

molecules, according to the energy scale. For notational simplicity $\psi^\dagger, \psi$ denote the whole set $\psi_j^\dagger(x), \psi_j(x)$ of creation-annihilation operators, i.e. one couple per particle species and spin component. This Hamiltonian includes the usual electromagnetic interactions accounted for in atomic and molecular physics. To incorporate gravitational interactions including self-interactions, we introduce complementary creation-annihilation operators $\chi_j^\dagger(x), \chi_j(x)$ and the overall Hamiltonian

$$H_G = H[\psi^\dagger, \psi] + H[\chi^\dagger, \chi] - G \sum_{j,k} m_j m_k \int dx dy \frac{\psi_j^\dagger(x)\psi_j(x)\chi_k^\dagger(y)\chi_k(y)}{|x-y|}, \tag{1}$$

acting on the tensor product $F_\psi \otimes F_\chi$ of the Fock spaces of the ψ and χ operators, where m_i denotes the mass of the i-th particle species and G is the gravitational constant. While the χ operators are taken to obey the same statistics as the original operators ψ, we take advantage of the arbitrariness pertaining to distinct operators and, for simplicity, we choose them commuting with one another: $[\psi, \chi]_- = [\psi, \chi^\dagger]_- = 0$.

The metaparticle state space S is identified with the subspace of $F_\psi \otimes F_\chi$ including the metastates obtained from the vacuum $|0\rangle = |0\rangle_\psi \otimes |0\rangle_\chi$ by applying operators built in terms of the products $\psi_j^\dagger(x)\chi_j^\dagger(y)$ and symmetrical with respect to the interchange $\psi^\dagger \leftrightarrow \chi^\dagger$, which, as a consequence, have the same number of ψ (green) and χ (red) metaparticles of each species. This is a consistent definition since the overall Hamiltonian is such that the corresponding time evolution is a group of (unitary) endomorphisms of S. If we prepare a pure n-particle state, represented in the original setting - excluding gravitational interactions - by

$$|g\rangle \doteq \int d^n x g(x_1, x_2, ..., x_n)\psi_{j_1}^\dagger(x_1)\psi_{j_2}^\dagger(x_2)...\psi_{j_n}^\dagger(x_n)|0\rangle, \tag{2}$$

its representation in S is given by the metastate

$$||g \otimes g\rangle\rangle = \int d^n x d^n y g(x_1, ..., x_n) g(y_1, ..., y_n)\psi_{j_1}^\dagger(x_1)...\psi_{j_n}^\dagger(x_n)\chi_{j_1}^\dagger(y_1)...\chi_{j_n}^\dagger(y_n)|0\rangle. \tag{3}$$

As for the physical algebra, it is identified with the operator algebra of say the green meta-world. In view of this, expectation values can be evaluated by preliminarily tracing out the χ operators and then taking the average in accordance with the traditional setting.

While we are talking trivialities as to an initial metastate like in Eq. (3), that is not the case in the course of time, since the overall Hamiltonian produces entanglement between the two metaworlds, leading, once χ operators are traced out, to mixed states of the physical algebra. It was shown in Ref. [1] that the ensuing non-unitary evolution induces both an effective interaction mimicking gravitation, and wave function localization. Localization is kept in time, as the spreading of the probability density of the center of mass of a macroscopic body does not imply a spreading of the wave function, but rather it is due to the emergence of a delocalized ensemble of localized pure states [13]. A peculiar feature of the model is that it cannot be obtained by quantizing its naive classical version, since the classical states corresponding to the constraint in $F_\psi \otimes F_\chi$, selecting the metastate space S, have green and red partners sitting in the same space point and then a divergent gravitational energy. While it is usual that, in passing from the classical to the quantum description, self-energy divergences are mitigated, in this instance we pass from a completely meaningless classical theory to a quite divergence free one. This is more transparent below, where we consider the field theoretic description.

Let us adopt here an interaction representation, where the free Hamiltonian is identified with $H[\psi^\dagger,\psi]+H[\chi^\dagger,\chi]$ and the time evolution of an initially untangled metastate $\left|\left|\tilde{\Phi}(0)\right\rangle\right\rangle$ is represented by

$$\begin{aligned}\left|\left|\tilde{\Phi}(t)\right\rangle\right\rangle &= T\exp\left[\frac{i}{\hbar}Gm^2\int dt\int dxdy\frac{\psi^\dagger(x,t)\psi(x,t)\chi^\dagger(y,t)\chi(y,t)}{|x-y|}\right]\left|\left|\tilde{\Phi}(0)\right\rangle\right\rangle \\ &\equiv U(t)\left|\left|\tilde{\Phi}(0)\right\rangle\right\rangle \equiv U(t)\left|\Phi(0)\right\rangle_\psi \otimes \left|\Phi(0)\right\rangle_\chi . \end{aligned} \tag{4}$$

Then, by making use of a Stratonovich-Hubbard transformation [7], we can rewrite the time evolution operator in the form

$$\begin{aligned}U(t) = &\int D\left[\varphi_1\right] D\left[\varphi_2\right] \exp\frac{ic^2}{2\hbar}\int dtdx\left[\varphi_1\nabla^2\varphi_1 - \varphi_2\nabla^2\varphi_2\right] \\ &T\exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G}\int dtdx\left[\varphi_1(x,t)+\varphi_2(x,t)\right]\psi^\dagger(x,t)\psi(x,t)\right] \\ &T\exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G}\int dtdx\left[\varphi_1(x,t)-\varphi_2(x,t)\right]\chi^\dagger(x,t)\chi(x,t)\right], \end{aligned} \tag{5}$$

namely as a functional integral over two auxiliary real scalar fields φ_1 and φ_2.

To give a physical interpretation of this result, consider the minimal variant of the Newton interaction in Eq. (4) aiming at avoiding instantaneous action at a distance, namely consider replacing $-1/|x-y|$ by the Feynman propagator $4\pi\Box^{-1} \equiv 4\pi\left(-\partial_t^2/c^2+\nabla^2\right)^{-1}$. Then the analog of Eq. (5) holds with the d'alembertian $\Box$ replacing the Laplacian ∇^2 and the ensuing expression can be read as the mixed path integral and operator expression for the evolution operator corresponding to the field Hamiltonian

$$H_{Field} = H[\psi^\dagger,\psi] + H[\chi^\dagger,\chi] + \frac{1}{2}\int dx\, \left[\pi_1^2 + c^2\left|\nabla\varphi_1\right|^2 - \pi_2^2 - c^2\left|\nabla\varphi_2\right|^2\right] \tag{6}$$

$$+mc\sqrt{2\pi G}\int dx\, \left\{\left[\varphi_1+\varphi_2\right]\psi^\dagger\psi + \left[\varphi_1-\varphi_2\right]\chi^\dagger\chi\right\}, \tag{7}$$

where $\pi_1 = \dot{\varphi}_1$ and $\pi_2 = \dot{\varphi}_2$ respectively denote the conjugate fields of φ_1 and φ_2 and all fields denote quantum operators. This theory can be read in analogy with nonrelativistic quantum electrodynamics, where a relativistic field is coupled with nonrelativistic matter, while the procedure to obtain the corresponding action at a distance theory by integrating out the φ fields is the analog of the Feynman's elimination of electromagnetic field variables [8].

The resulting theory, containing the negative energy field φ_2, has the attractive feature of being divergence free, at least in the non-relativistic limit, where Feynman graphs with virtual particle-antiparticle pairs can be omitted. To be specific, it does not require the infinite self-energy subtraction needed for instance in electrodynamics on evaluating the Lamb shift, or the coupling constant renormalization [9]. Here of course we refer to the covariant perturbative formalism applied to our model, where matter fields are replaced by their relativistic counterparts and the non-relativistic character of the model is reflected in the mass density being considered as a scalar coupled with the scalar fields by Yukawa-like interactions. In fact there is a complete cancellation among all Feynman diagrams containing only ψ (or equivalently χ) and internal φ lines, owing to the difference in sign between the φ_1 and the φ_2 free propagators. This state of affairs of course is the field theoretic counterpart of the absence of direct $\psi-\psi$ and $\chi-\chi$ interactions in the theory obtained

by integrating out the φ operators, whose presence would otherwise require the infinite self-energy subtraction corresponding to normal ordering. These considerations, supported by the mentioned suggestions derived from a phenomenological analysis [12] about the possible role of negative energy fields, may provide substantial clues for the possible extensions of the model towards a relativistic theory of gravity-induced localization. On the other hand a Yukawa interaction with a (positive energy) scalar field emerges also moving from Einstein's theory of gravitation, if one confines consideration to conformal space-time fluctuations in a linear approximation [14,15].

Going back to our evolved metastate (4), the corresponding physical state is given by

$$M(t) \equiv Tr_\chi \left|\left|\tilde{\Phi}(t)\right\rangle\right\rangle \left\langle\left\langle\tilde{\Phi}(t)\right|\right| = \sum_k \quad {}_\chi\langle k| \left|\left|\tilde{\Phi}(t)\right\rangle\right\rangle \left\langle\left\langle\tilde{\Phi}(t)\right|\right| |k\rangle_\chi , \tag{8}$$

and, by using Eq. (5), we can write

$$\begin{aligned}
{}_\chi\langle k| \left|\left|\tilde{\Phi}(t)\right\rangle\right\rangle = & \int D[\varphi_1] D[\varphi_2] \exp \frac{ic^2}{2\hbar} \int dtdx \left[\varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2\right] \\
& {}_\chi\langle k| T \exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1(x,t) - \varphi_2(x,t)\right] \chi^\dagger(x,t)\chi(x,t)\right] |\Phi(0)\rangle_\chi \\
& T \exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1(x,t) + \varphi_2(x,t)\right] \psi^\dagger(x,t)\psi(x,t)\right] |\Phi(0)\rangle_\psi .
\end{aligned} \tag{9}$$

Then the final expression for the physical state at time t is given by

$$M(t) = \tag{10}$$

$$\begin{aligned}
& \int D[\varphi_1] D[\varphi_2] D[\varphi_1'] D[\varphi_2'] \exp \frac{ic^2}{2\hbar} \int dtdx \left[\varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 - \varphi_1' \nabla^2 \varphi_1' + \varphi_2' \nabla^2 \varphi_2'\right] \\
& {}_\chi\langle\Phi(0)| T^{-1} \exp\left[i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1' - \varphi_2'\right] \chi^\dagger\chi\right] T \exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1 - \varphi_2\right] \chi^\dagger\chi\right] |\Phi(0)\rangle_\chi \\
& T \exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1 + \varphi_2\right] \psi^\dagger\psi\right] |\Phi(0)\rangle_{\psi\psi} \langle\Phi(0)| T^{-1} \exp i\frac{mc}{\hbar}\sqrt{2\pi G} \int dtdx \left[\varphi_1' + \varphi_2'\right] \psi^\dagger\psi,
\end{aligned}$$

where, due to the constraint on the metastate space, χ operators can be replaced by ψ operators, if simultaneously the metastate vector $|\Phi(0)\rangle_\chi$ is replaced by $|\Phi(0)\rangle_\psi$. Then, if in the c-number factor corresponding to the χ-trace, we make the mean field (MF) approximation $\psi^\dagger\psi \to \langle\psi^\dagger\psi\rangle$, we get

$$M_{MF}(t) =$$

$$\int D\left[\varphi_1\right] D\left[\varphi_2\right] D\left[\varphi_1'\right] D\left[\varphi_2'\right] \exp \frac{ic^2}{2\hbar} \int dtdx \left[\varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 - \varphi_1' \nabla^2 \varphi_1' + \varphi_2' \nabla^2 \varphi_2'\right]$$
$$T \exp \left[-i \frac{mc}{\hbar} \sqrt{2\pi G} \int dtdx \left[\varphi_1 \left[\psi^\dagger \psi + \left\langle \psi^\dagger \psi \right\rangle\right] + \varphi_2 \left[\psi^\dagger \psi - \left\langle \psi^\dagger \psi \right\rangle\right]\right]\right] \left|\Phi(0)\right\rangle_\psi$$
$$_\psi \left\langle \Phi(0)\right| T^{-1} \exp i \frac{mc}{\hbar} \sqrt{2\pi G} \int dtdx \left[\varphi_1' \left[\psi^\dagger \psi + \left\langle \psi^\dagger \psi \right\rangle\right] + \varphi_2' \left[\psi^\dagger \psi - \left\langle \psi^\dagger \psi \right\rangle\right]\right]. \tag{11}$$

Finally, if we perform functional integrations, we get

$$M_{MF}(t) =$$
$$T \exp \left[\frac{i}{\hbar} G m^2 \int dt \int dxdy \frac{\psi^\dagger(x,t)\psi(x,t) \left\langle \psi^\dagger(y,t)\psi(y,t)\right\rangle}{|x-y|}\right] \left|\Phi(0)\right\rangle_\psi$$
$$_\psi \left\langle \Phi(0)\right| T^{-1} \exp \left[-\frac{i}{\hbar} G m^2 \int dt \int dxdy \frac{\psi^\dagger(x,t)\psi(x,t) \left\langle \psi^\dagger(y,t)\psi(y,t)\right\rangle}{|x-y|}\right], \tag{12}$$

namely in this approximation the model is equivalent to the S-N theory [6]. Of course the whole procedure could be repeated without substantial variations starting from the field theoretic Hamiltonian (6), inserting the mean field approximation before applying the Feynman's procedure for the elimination of field variables, and then getting a retarded potential version of the S-N model.

However this approximation does not share with the original model the crucial ability of making linear superpositions of macroscopically different states unobservable. Consider in fact an initial state corresponding to the linear, for simplicity orthogonal, superposition of N localized states of one and the same macroscopic body, which were shown to exist as pure states corresponding to unentangled bound metastates of green and red metamatter for bodies of ordinary density and a mass higher than $\sim 10^{11}$ proton masses [1]:

$$\left|\Phi(0)\right\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left|z_j\right\rangle, \tag{13}$$

where $|z\rangle$ represents a localized state centered in z. Compare the coherence $\langle z_h| M(t) |z_k\rangle$ when evaluated according to Eqs. (10) and (12), where we consider the localized states as approximate eigenstates of the particle density operator

$$\psi^\dagger(x,t)\psi(x,t)\left|z\right\rangle \simeq n(x-z)\left|z\right\rangle \tag{14}$$

and time dependence in $\psi^\dagger\psi$ irrelevant, as the considered states are stationary states in the Schroedinger picture apart from an extremely slow spreading [13].

According to the original model we get, from Eq. (10)

$$\begin{aligned}
&\left\langle z_h\right| M(t)\left|z_k\right\rangle \\
&= \int D\left[\varphi_1\right] D\left[\varphi_2\right] D\left[\varphi_1'\right] D\left[\varphi_2'\right] \exp\frac{ic^2}{2\hbar}\int dtdx\left[\varphi_1\nabla^2\varphi_1-\varphi_2\nabla^2\varphi_2-\varphi_1'\nabla^2\varphi_1'+\varphi_2'\nabla^2\varphi_2'\right] \\
&\quad \frac{1}{N^2}\sum_{j=1}^{N}\exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G}\int dtdx\left[\left[\varphi_1-\varphi_2\right]n(x-z_j)-\left[\varphi_1'-\varphi_2'\right]n(x-z_j)\right)\right] \\
&\quad \exp\left[-i\frac{mc}{\hbar}\sqrt{2\pi G}\int dtdx\left[\left[\varphi_1+\varphi_2\right]n(x-z_h)-\left[\varphi_1'+\varphi_2'\right]n(x-z_k)\right)\right],
\end{aligned} \tag{15}$$

and, after integrating out the scalar fields,

$$\left\langle z_h\right| M(t)\left|z_k\right\rangle = \frac{1}{N^2}\sum_{j=1}^{N}\exp\frac{i}{\hbar}Gm^2t\int dxdy\left[\frac{n(x-z_j)n(y-z_h)}{|x-y|}-\frac{n(x-z_j)n(y-z_k)}{|x-y|}\right], \tag{16}$$

which shows that, while diagonal coherences are given by $\left\langle z_h\right| M(t)\left|z_h\right\rangle = 1/N$, the off-diagonal ones, under reasonable assumptions on the linear superposition in Eq. (13) of a large number of localized states, approximately vanish, due to the random phases in the sum in Eq. (16). This makes the state $M(t)$, for not too short times, equivalent to an ensemble of localized states:

$$M(t) \simeq \frac{1}{N}\sum_{j=1}^{N}\left|z_j\right\rangle\left\langle z_j\right|. \tag{17}$$

On the other hand, if we calculate coherences according to the S-N model, we get

$$\begin{aligned}
&\left\langle z_h\right| M_{MF}(t)\left|z_k\right\rangle \\
&= \frac{1}{N}\exp\frac{i}{\hbar}Gm^2t\int dxdy\frac{\left[n(x-z_h)-n(x-z_k)\right]\sum_{j=1}^{N}n(y-z_j)/N}{|x-y|},
\end{aligned} \tag{18}$$

so that here the sum appears in the exponent and there is no cancellation. While diagonal coherences keep the same value as in the original model, off-diagonal ones acquire only a

phase for the presence of the mean gravitational interaction, but keep the same absolute value $1/N$ as the diagonal ones. Furthermore, if we take $N = 2$ rather than N very large, the S-N approximation gives just $\langle z_1| \, M_{MF}(t) \, |z_2\rangle = 1/2$, whereas the exact model still offers a mechanism to make off-diagonal coherences unobservable, due to time oscillations [1].

It is worth while to remind that, although both our proposal and the S-N model give rise to localized states and reproduce the classical aspects of the gravitational interaction, the mean field approximation, necessary to pass from the former to the latter, spoils the theory, not only of its feature of reducing unlocalized wave functions, but also of another desirable property. In fact it can be shown that according to our model localized states evolve into unlocalized ensembles of localized states [13], while the S-N theory leads to stationary localized states [6], which is rather counterintuitive and unphysical, since space localization implies linear momentum uncertainty, and this, in its turn, should imply a spreading of the probability distribution in space.

In conclusion, while the present model has only a nonrelativistic character, its analysis hints of possible directions for extensions to higher energies, where an instantaneous action at a distance is not appropriate. In particular, the emergence of negative energy fields leads naturally to a promising perspective for the construction of finite field theories, where divergence cancellations are due to the presence of couples of positive and negative energy fields, rather than of supersymmetric partners. Furthermore, since the geometric formulation of Newtonian gravity, i.e. the Newton-Cartan theory, leads only to the mean field approximation, i.e. the S-N theory [16], it may be likely that the geometric aspects of gravity may even play a misleading role in looking for a quantum theory including gravity both in its classical aspects and in its possible localization effects. More specifically the Einstein theory of gravitation could arise, unlike, for instance, classical electrodynamics, not as a result of taking expectation values with respect to a pure physical state, but rather, as an effective long distance theory like hydrodynamics, from a statistical average, or equivalently by tracing out unobservable degrees of freedom starting from a pure metastate.

Acknowledgments - Financial support from M.U.R.S.T., Italy and I.N.F.M., Salerno is

acknowledged

---

* Phone: +39 89 965229; FAX: +39 89 965275; electronic address: defilippo@sa.infn.it

[1] S. De Filippo, *Emergence of classicality in non relativistic quantum mechanics through gravitational self-interaction*, LANL arXiv:quant-ph/0104052.

[2] F. Karolyhazy, A. Frenkel and B. Lukacs B., in *Quantum Concepts in Space and Time,* R. Penrose and C.J. Isham Eds. Clarenden, Oxford Science Publications, Oxford (1986), refs. therein.

[3] L. Diosi, Phys. Lett. A **105**, 199 (1984).

[4] L. Diosi, Phys. Rev. A **40**, 1165 (1989), refs. therein.

[5] R. Penrose, Gen. Rel. Grav. **28**, 581 (1996).

[6] D. Kumar, V. Soni, Phys. Lett. A **271**, 157 (2000).

[7] J.W. Negele, H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley P.C., NY, 1988).

[8] R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[9] C. Itzykson, J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[10] P. Pearle, E. Squires, Phys. Rev. Lett. **73** 1 (1994).

[11] P. Pearle, J. Ring, J. I. Collar and T. Frank, Found. Phys. **29**, 465 (1999).

[12] P. Pearle, Phy. Rev. A 48, 913 (1993).

[13] S. De Filippo, *Gravity-induced entropy in the quantum motion of a macroscopic body*, LANL arXiv:quant-ph/0105013.

[14] J.L. Rosales, J.L. Sanchez-Gomez, Phys. Lett. A **199**, 320 (1995).

[15] W.L. Power, I. C. Percival, Proc. Roy. Soc. Lond. **A456**, 955 (2000).

[16] J. Christian, Phys. Rev. D **56**, 4844 (1997).